東京大学本郷キャンパス 工学部8号館 83講義室 (地下1階)

FrontISTRの非圧縮性流体解析機能 (機能説明および理論解説)

東京大学
新領域創成科学研究科
人間環境学専攻
橋本　学

2017年2月27日
第34回FrontISTR研究会
＜広がり続けるFrontISTRの適用分野＞

FrontISTRの非圧縮性流体解析機能
(機能説明および理論解説)

発表内容

1. 機能開発の目的
2. FrontISTR Ver. 4.6での機能
3. SUPG/PSPGによる定式化
4. 今後の展開

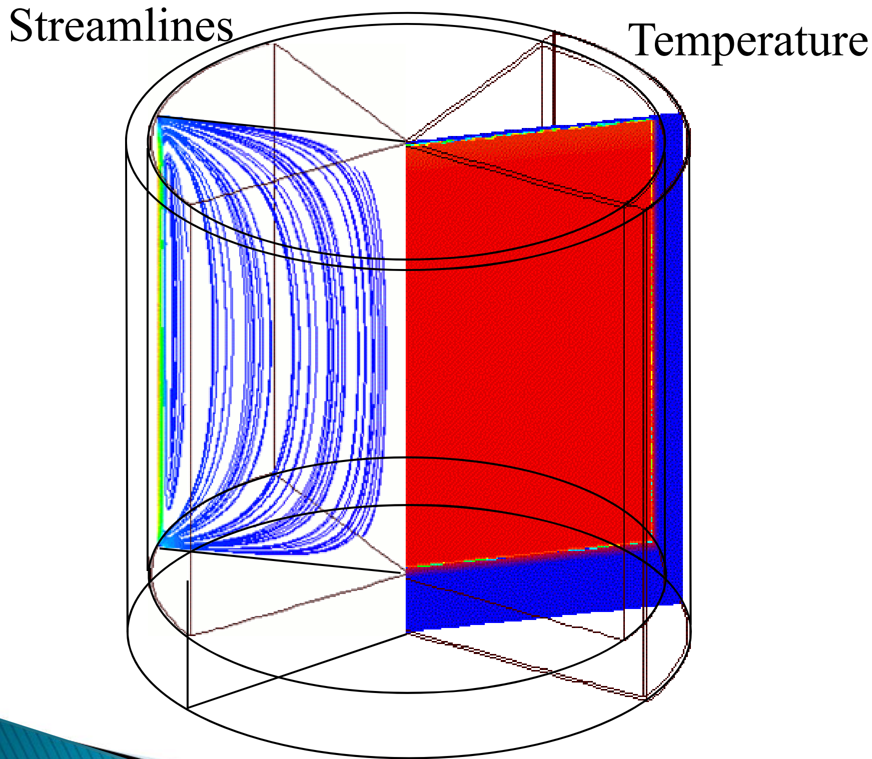
FrontISTRの非圧縮性流体解析機能
(機能説明および理論解説)

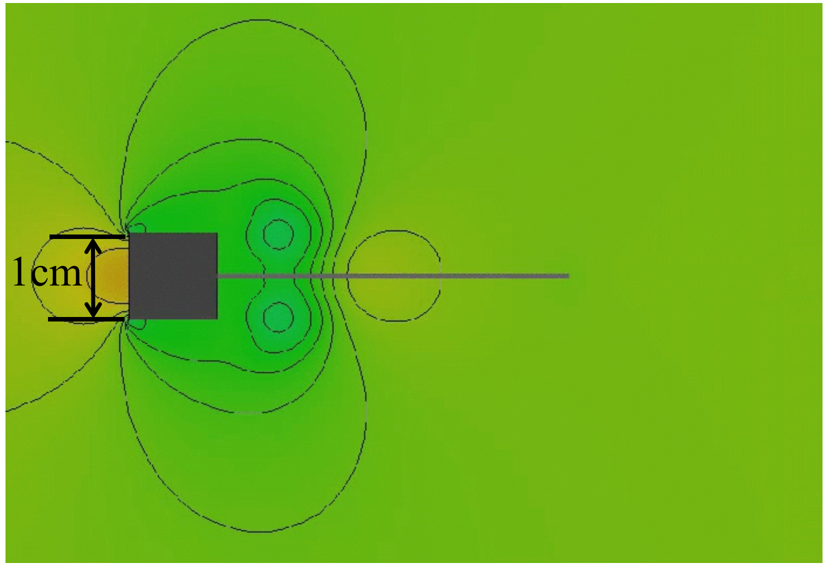
発表内容

1. 機能開発の目的
2. FrontISTR Ver. 4.6での機能
3. SUPG/PSPGによる定式化
4. 今後の展開

機能開発の目的

構造周囲の流れが構造の変形に影響する現象を解析可能にする





Pressure contours at *Re* = 332 until *t* = 6.0 s

対流伝熱が影響する
高温構造の熱変形

流体トラクションによる
薄肉構造の大変形

FrontISTRの非圧縮性流体解析機能 (機能説明および理論解説)

発表内容

1. 機能開発の目的
2. FrontISTR Ver. 4.6での機能
3. SUPG/PSPGによる定式化
4. 今後の展開

FrontISTR Ver.4.6での機能

- 非圧縮性流れの有限要素定式化として，SUPG/PSPG[1][2]を採用
  要素剛性マトリックスの作成，全体剛性マトリックスの作成，
  連立一次方程式の求解というFrontISTRプログラムへの実装が容易
- 境界条件は，流速既定，圧力規定，トラクションフリー
- 体積力として，重力を与えることも可能

※ 未対応の機能
温度の移流拡散方程式 (熱エネルギー方程式)，自然対流，
気液二相流，Fluid-Structure Interaction，乱流，非Newton流体など

[1] Tezduyar, T.E. *Advanced in Applied Mechanics*, Vol.28, pp.1-44, 1991.
[2] Tezduyar, T.E., Mittal, S., Ray, S.E. and Shih, R. *Computer Methods in Applied Mechanics and Engineering*, Vol.95, pp.221-242, 1992.

FrontISTRの非圧縮性流体解析機能 (機能説明および理論解説)

発表内容

1. 機能開発の目的
2. FrontISTR Ver. 4.6での機能
3. SUPG/PSPGによる定式化
4. 今後の展開

流体の支配方程式系 (1/2)

- 流れは非圧縮性
- 構成則はNewton流体

(連続の式)

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{in } V$$

(運動方程式)

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{b}$$

$$\boldsymbol{\tau} = \mu \left\{ \nabla \otimes \boldsymbol{v} + (\nabla \otimes \boldsymbol{v})^{\mathrm{T}} \right\}$$

$$\text{in } V$$

(境界条件)

$$\begin{cases} \boldsymbol{v} = \underline{\boldsymbol{v}} & \text{on } S_v \\ \boldsymbol{t} = \underline{\boldsymbol{t}} & \text{on } S_t \end{cases}$$

$\boldsymbol{v}$: 速度 [m/s] 未知量

p : 圧力 [Pa] 未知量

∇ : ナブラ [1/m]

$$\boxed{\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z}}$$

ρ : 密度 [kg/m^3]

$\partial / \partial t$: 固定時間微分 [1/s]

$\boldsymbol{\tau}$: 粘性応力 [Pa]

$\boldsymbol{b}$: 単位質量当たりの体積力 [m/s^2]

μ : 粘性係数 [Pa・s]

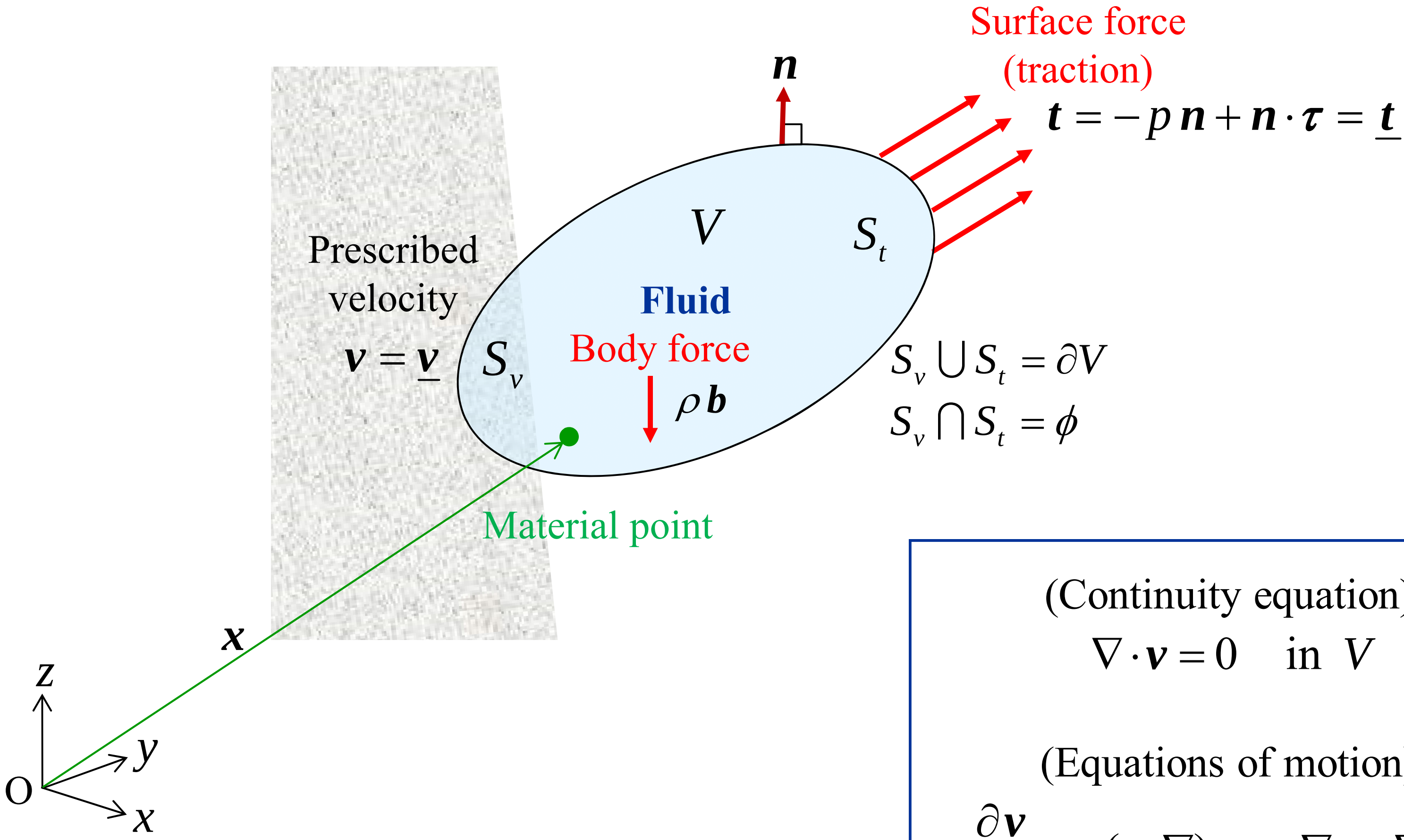
T : 転置記号

$\boldsymbol{t}$: トラクションベクトル [Pa]

$$\boxed{\boldsymbol{t} = -p\,\boldsymbol{n} + \boldsymbol{n} \cdot \boldsymbol{\tau}}$$

$\boldsymbol{n}$: 外向き単位法線ベクトル [-]

流体の支配方程式系 (2/2)



(Continuity equation)

$$\nabla\cdot\boldsymbol{v} = 0 \quad \text{in } V$$

(Equations of motion)

$$\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \nabla\cdot\boldsymbol{\tau} + \rho\boldsymbol{b}$$

$$\boldsymbol{\tau} = \mu\left\{\nabla\otimes\boldsymbol{v} + (\nabla\otimes\boldsymbol{v})^{\mathrm{T}}\right\}$$

in V

SUPG/PSPG (1/10)

Galerkin Least Squares法 (安定化有限要素法) による定式化
(最終的には，SUPG/PSPG法が導出)

支配方程式

$$\begin{cases} \rho \dfrac{\partial \boldsymbol{v}}{\partial t} + \rho (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{b} \\ \nabla \cdot \boldsymbol{v} = 0 \end{cases}$$

近似解を $\hat{\boldsymbol{v}}$, $\hat{p}$ とすると

$$\begin{cases} \rho \dfrac{\partial \hat{\boldsymbol{v}}}{\partial t} + \rho (\hat{\boldsymbol{v}} \cdot \nabla) \hat{\boldsymbol{v}} + \nabla \hat{p} - \nabla \cdot \hat{\boldsymbol{\tau}} - \rho \boldsymbol{b} = \boldsymbol{r} & \text{残差} \\ \nabla \cdot \hat{\boldsymbol{v}} = q & \text{残差} \end{cases}$$

ただし，$\hat{\boldsymbol{v}} = \underline{\boldsymbol{v}}$ on S_v , $\hat{\boldsymbol{t}} = \underline{\boldsymbol{t}}$ on S_t

$$R = \frac{1}{2} \int_V \boldsymbol{r} \cdot \boldsymbol{r} \, dV$$

最小となる近似解を求める

停留条件

$$\frac{\partial R}{\partial \boldsymbol{v}^I} = \boldsymbol{0}, \quad \frac{\partial R}{\partial p^I} = 0$$

[1] Tezduyar, T.E. *Advanced in Applied Mechanics*, Vol.28, pp.1-44, 1991.
[2] Tezduyar, T.E., Mittal, S., Ray, S.E. and Shih, R. *Computer Methods in Applied Mechanics and Engineering*, Vol.95, pp.221-242, 1992.

SUPG/PSPG (2/10)

FrontISTR Ver.4.6では4節点四面体要素のみ実装

物理空間座標

$$\boldsymbol{x} = \sum_{\alpha} N^{(\alpha)}(\xi)\ \boldsymbol{x}^{(\alpha)} \qquad (\alpha):\ \text{local node number}$$

$$= \sum_{I} N^{I}(\xi)\ \boldsymbol{x}^{I} \qquad I:\ \text{global node number}$$

速度

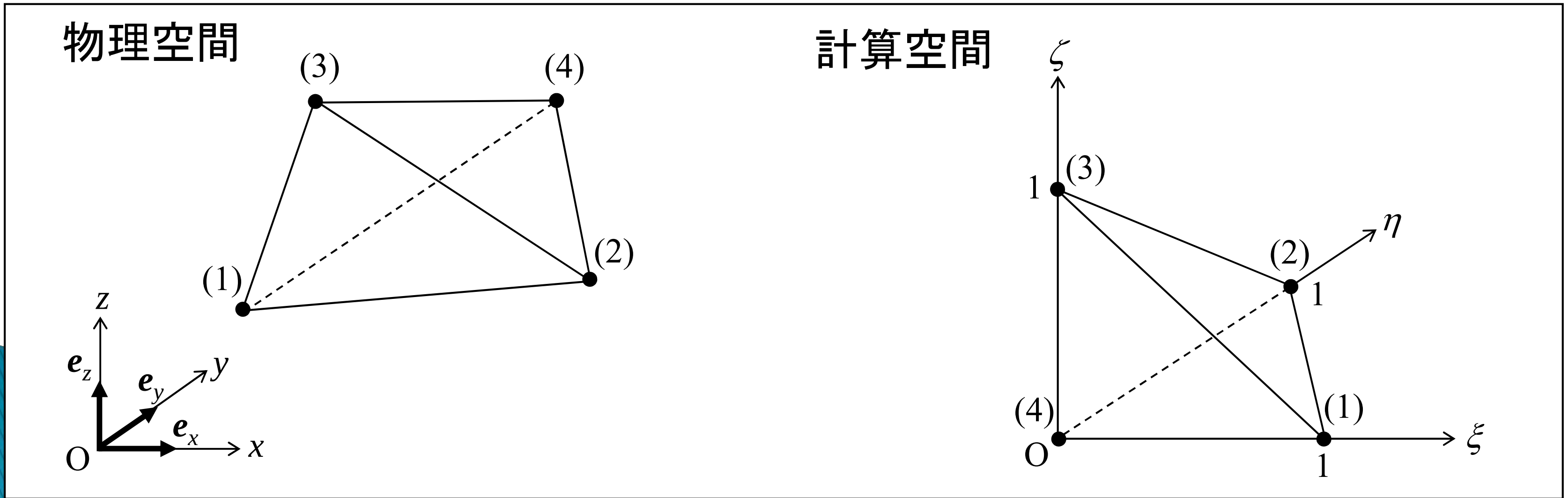
$$\hat{\boldsymbol{v}} = \sum_{I} N^{I}(\xi)\ \boldsymbol{v}^{I} = \sum_{\alpha} N^{(\alpha)}(\xi)\ \boldsymbol{v}^{(\alpha)}$$

圧力

$$\hat{p} = \sum_{I} N^{I}(\xi)\ p^{I} = \sum_{\alpha} N^{(\alpha)}(\xi)\ p^{(\alpha)}$$

形状関数

$$\begin{cases} N^{(1)}(\xi) = \xi \\ N^{(2)}(\xi) = \eta \\ N^{(3)}(\xi) = \zeta \\ N^{(4)}(\xi) = 1 - \xi - \eta - \zeta \end{cases}$$



SUPG/PSPG (3/10)

$$\frac{\partial R}{\partial v_i^I}=0$$

$$0=\frac{\partial}{\partial v_i^I}\left(\frac{1}{2}\int_V r_k\, r_k\, dV\right)$$

$$=\int_V \frac{\partial r_k}{\partial v_i^I}\, r_k\, dV = 0$$

$$=\int_V \frac{\partial}{\partial v_i^I}\left\{\rho\frac{\partial \hat{v}_k}{\partial t}+\rho\hat{v}_j\, \hat{v}_{k,j}+\hat{p}_{,k}-\mu\,(\hat{v}_{k,jj}+\hat{v}_{j,kj})-\rho b_k\right\} r_k\, dV$$

$$=\sum_e\left[\int_{V^e}\frac{\partial}{\partial v_i^{(\alpha)}}\left\{\rho\frac{\partial \hat{v}_k}{\partial t}+\rho\underline{\bar{v}_j^e}\, \hat{v}_{k,j}+\hat{p}_{,k}-\mu\,(\hat{v}_{k,jj}+\hat{v}_{j,kj})-\rho b_k\right\} r_k\, dV\right]$$

$$=\sum_e\left[\int_{V^e}\frac{\partial}{\partial v_i^{(\alpha)}}\left\{\rho\cancel{\frac{\partial N^{(\beta)}}{\partial t}}v_k^{(\beta)}+\rho\bar{v}_j^e\, N_{,j}^{(\beta)}\, v_k^{(\beta)}-\mu\,(N_{,jj}^{(\beta)}\, v_k^{(\beta)}+N_{,kj}^{(\beta)}\, v_j^{(\beta)})-\rho b_k\right\} r_k\, dV\right]$$

$$=\sum_e\left[\int_{V^e}\left\{\rho\,\bar{v}_j^e\, N_{,j}^{(\alpha)}\,\delta_{ki}-\mu(\cancel{N_{,jj}^{(\alpha)}}\,\delta_{ki}+\cancel{N_{,kj}^{(\alpha)}}\delta_{ji})\right\} r_k\, dV\right]$$

4節点四面体要素の場合

$$\bar{v}_i^e=\frac{1}{V^e}\int_{V^e}\hat{v}_i\, dV$$

：速度の要素平均値 [m/s]

I : global node number

$\leftrightarrow$ (α): local node number

$$0=\sum_e\left\{\sum_\alpha\left(\int_{V^e}\delta v_i^{(\alpha)}\rho\,\bar{v}_j^e\, N_{,j}^{(\alpha)}\,\delta_{ki}\, r_k\, dV\right)\right\}$$

ただし，$\delta v_i^{(\alpha)}=0$　on S_v

$$=\sum_e\left(\int_{V^e}\rho\,\bar{v}_j^e\,\delta v_{i,j}\, r_i\, dV\right)$$

$$=\int_V\rho\,\hat{v}_j\,\delta v_{i,j}\, r_i\, dV$$

SUPG/PSPG (4/10)

$$\frac{\partial R}{\partial p^I} = 0$$

$$0 = \frac{\partial}{\partial p^I}\left(\frac{1}{2}\int_V r_k\, r_k\, dV\right)$$

$$= \int_V \frac{\partial r_k}{\partial p^I}\, r_k\, dV$$

$$= \int_V \frac{\partial}{\partial p^I}\left\{\rho \frac{\partial \hat{v}_k}{\partial t} + \rho\, \hat{v}_j\, \hat{v}_{k,j} + \hat{p}_{,k} - \mu\,(\hat{v}_{k,jj} + \hat{v}_{j,kj}) + \rho\, b_k\right\} r_k\, dV$$

$$= \sum_e \left[\int_{V^e} \frac{\partial}{\partial p^{(\alpha)}}\left\{\rho \frac{\partial \hat{v}_k}{\partial t} + \rho\, \overline{v}_j^e\, \hat{v}_{k,j} + \hat{p}_{,k} - \mu\,(\hat{v}_{k,jj} + \hat{v}_{j,kj}) + \rho\, b_k\right\} r_k\, dV\right]$$

$$= \sum_e \left(\int_{V^e} N_{,k}^{(\alpha)}\, r_k\, dV\right)$$

$$0 = \sum_e \left\{\sum_\alpha \left(\int_{V^e} \delta p^{(\alpha)} N_{,i}^{(\alpha)}\, r_i\, dV\right)\right\}$$

$$= \sum_e \left(\int_{V^e} \delta p_{,i}\, r_i\, dV\right)$$

$$= \int_V \delta p_{,i}\, r_i\, dV$$

SUPG/PSPG (5/10)

$$
\begin{cases}
\int_V \delta \boldsymbol{v} \cdot \boldsymbol{r}\, dV + \int_V \dfrac{\tau}{\rho} \rho (\hat{\boldsymbol{v}} \cdot \nabla) \delta \boldsymbol{v} \cdot \boldsymbol{r}\, dV = 0 \\
\int_V \delta p\ q\, dV + \int_V \dfrac{\tau_p}{\rho} \nabla \delta p \cdot \boldsymbol{r}\, dV = 0
\end{cases}
$$

τ, τ_p ：安定化パラメータ [s]

$$
\begin{cases}
\int_V \delta \boldsymbol{v} \cdot \left[\rho \dfrac{\partial \hat{\boldsymbol{v}}}{\partial t} + \rho (\hat{\boldsymbol{v}} \cdot \nabla) \hat{\boldsymbol{v}} + \nabla \hat{p} - \mu \nabla \cdot \left\{ \nabla \otimes \hat{\boldsymbol{v}} + (\nabla \otimes \hat{\boldsymbol{v}})^{\mathrm{T}} \right\} - \rho \boldsymbol{b} \right] dV \\
\quad + \int_V \tau (\hat{\boldsymbol{v}} \cdot \nabla) \delta \boldsymbol{v} \cdot \left[\rho \dfrac{\partial \hat{\boldsymbol{v}}}{\partial t} + \rho (\hat{\boldsymbol{v}} \cdot \nabla) \hat{\boldsymbol{v}} + \nabla \hat{p} - \cancel{\mu \nabla \cdot \left\{ \nabla \otimes \hat{\boldsymbol{v}} + (\nabla \otimes \hat{\boldsymbol{v}})^{\mathrm{T}} \right\}} - \rho \boldsymbol{b} \right] dV = 0 \\
\int_V \delta p \nabla \cdot \hat{\boldsymbol{v}}\, dV \\
\quad + \int_V \dfrac{\tau_p}{\rho} \nabla \delta p \cdot \left[\rho \dfrac{\partial \hat{\boldsymbol{v}}}{\partial t} + \rho (\hat{\boldsymbol{v}} \cdot \nabla) \hat{\boldsymbol{v}} + \nabla \hat{p} - \cancel{\mu \nabla \cdot \left\{ \nabla \otimes \hat{\boldsymbol{v}} + (\nabla \otimes \hat{\boldsymbol{v}})^{\mathrm{T}} \right\}} - \rho \boldsymbol{b} \right] dV = 0
\end{cases}
$$

4節点四面体要素の場合

4節点四面体要素の場合

SUPG/PSPG (6/10)

$$\int_V \delta\boldsymbol{v}\cdot\left\{\rho\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}+\nabla\hat{p}-\nabla\cdot\hat{\boldsymbol{\tau}}-\rho\boldsymbol{b}\right\}dV$$

$$=\int_V \delta\boldsymbol{v}\cdot\left\{\rho\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\right\}dV+\int_V \delta\boldsymbol{v}\cdot\left(\nabla\hat{p}-\nabla\cdot\hat{\boldsymbol{\tau}}-\rho\boldsymbol{b}\right)dV$$

$$=\int_V \delta\boldsymbol{v}\cdot\left\{\rho\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\right\}dV+\int_V \{\underbrace{\nabla\cdot(\delta\boldsymbol{v}\,\hat{p})-(\nabla\cdot\delta\boldsymbol{v})\,\hat{p}}_{\text{部分積分}}\underbrace{-\nabla\cdot(\hat{\boldsymbol{\tau}}\cdot\delta\boldsymbol{v})+\nabla\otimes\delta\boldsymbol{v}:\hat{\boldsymbol{\tau}}}_{\text{部分積分}}-\delta\boldsymbol{v}\cdot\rho\boldsymbol{b}\}\,dV$$

$$=\int_V \delta\boldsymbol{v}\cdot\left\{\rho\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\right\}dV+\int_V \nabla\cdot\{\delta\boldsymbol{v}\cdot(\hat{p}\,\boldsymbol{I})-\hat{\boldsymbol{\tau}}\cdot\delta\boldsymbol{v}\}\,dV-\int_V\{(\nabla\cdot\delta\boldsymbol{v})\,\hat{p}-\nabla\otimes\delta\boldsymbol{v}:\hat{\boldsymbol{\tau}}-\delta\boldsymbol{v}\cdot\rho\boldsymbol{b}\}\,dV$$

$$=\int_V \delta\boldsymbol{v}\cdot\left\{\rho\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\right\}dV-\underbrace{\oint_{\partial V}\boldsymbol{n}\cdot\{\delta\boldsymbol{v}\cdot(-\hat{p}\,\boldsymbol{I})+\hat{\boldsymbol{\tau}}\cdot\delta\boldsymbol{v}\}\,dS}_{\text{Gaussの発散定理}}-\int_V\{(\nabla\cdot\delta\boldsymbol{v})\,\hat{p}-\nabla\otimes\delta\boldsymbol{v}:\hat{\boldsymbol{\tau}}-\delta\boldsymbol{v}\cdot\rho\boldsymbol{b}\}\,dV$$

$$=\int_V\left[\rho\delta\boldsymbol{v}\cdot\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho\,\delta\boldsymbol{v}\cdot\{(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\}-(\nabla\cdot\delta\boldsymbol{v})\,\hat{p}+\nabla\otimes\delta\boldsymbol{v}:\hat{\boldsymbol{\tau}}-\delta\boldsymbol{v}\cdot\rho\boldsymbol{b}\right]dV-\oint_{\partial V}\delta\boldsymbol{v}\cdot(-\hat{p}\,\boldsymbol{n}+\boldsymbol{n}\cdot\hat{\boldsymbol{\tau}})\,dS$$

$$=\int_V\left[\rho\delta\boldsymbol{v}\cdot\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho\,\delta\boldsymbol{v}\cdot\{(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\}-(\nabla\cdot\delta\boldsymbol{v})\,\hat{p}+\nabla\otimes\delta\boldsymbol{v}:\hat{\boldsymbol{\tau}}-\delta\boldsymbol{v}\cdot\rho\boldsymbol{b}\right]dV-\oint_{\partial V}\delta\boldsymbol{v}\cdot\hat{\boldsymbol{t}}\,dS$$

$$=\int_V\left[\rho\delta\boldsymbol{v}\cdot\frac{\partial\hat{\boldsymbol{v}}}{\partial t}+\rho\,\delta\boldsymbol{v}\cdot\{(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}}\}-(\nabla\cdot\delta\boldsymbol{v})\,\hat{p}+\nabla\otimes\delta\boldsymbol{v}:\hat{\boldsymbol{\tau}}-\delta\boldsymbol{v}\cdot\rho\boldsymbol{b}\right]dV-\int_{S_t}\delta\boldsymbol{v}\cdot\underline{\boldsymbol{t}}\,dS$$

SUPG/PSPG (7/10)

$$
\begin{cases}
\sum_e \int_{V^e} \left[\rho (N^{(\alpha)} \delta \boldsymbol{v}^{(\alpha)}) \cdot \dfrac{\partial (N^{(\beta)} \boldsymbol{v}^{(\beta)})}{\partial t} + \rho (N^{(\alpha)} \delta \boldsymbol{v}^{(\alpha)}) \cdot \left\{ (\overline{\boldsymbol{v}}^e \cdot \nabla)(N^{(\beta)} \boldsymbol{v}^{(\beta)}) \right\} \right. \\
\qquad \left. - \left\{ \nabla \cdot (N^{(\alpha)} \delta \boldsymbol{v}^{(\alpha)}) \right\} (N^{(\beta)} p^{(\beta)}) + \mu \nabla \otimes (N^{(\alpha)} \delta \boldsymbol{v}^{(\alpha)}) : \left\{ \nabla \otimes (N^{(\beta)} \boldsymbol{v}^{(\beta)}) + (\nabla \otimes (N^{(\beta)} \boldsymbol{v}^{(\beta)})^{\mathrm{T}} \right\} - (N^{(\alpha)} \delta \boldsymbol{v}^{(\alpha)}) \cdot \rho \boldsymbol{b} \right] dV \\
\qquad - \int_{S_t} (N^{(\alpha)} \delta \boldsymbol{v}^{(\alpha)}) \cdot \underline{\boldsymbol{t}}\, dS \\
\qquad + \sum_e \int_{V^e} \tau (\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\alpha)}) \left\{ \rho \dfrac{\partial (N^{(\beta)} \boldsymbol{v}^{(\beta)})}{\partial t} + \rho (\overline{\boldsymbol{v}}^e \cdot \nabla)(N^{(\beta)} \boldsymbol{v}^{(\beta)}) + \nabla (N^{(\beta)} p^{(\beta)}) - \rho \boldsymbol{b} \right\} dV = 0 \\
\int_V (N^{(\alpha)} \delta p^{(\alpha)}) \nabla \cdot (N^{(\beta)} \boldsymbol{v}^{(\beta)})\, dV \\
\qquad + \int_V \dfrac{\tau_p}{\rho} \nabla (N^{(\alpha)} \delta p^{(\alpha)}) \cdot \left\{ \rho \dfrac{\partial (N^{(\beta)} \boldsymbol{v}^{(\beta)})}{\partial t} + \rho (\overline{\boldsymbol{v}}^e \cdot \nabla)(N^{(\beta)} \boldsymbol{v}^{(\beta)}) + \nabla (N^{(\beta)} p^{(\beta)}) - \rho \boldsymbol{b} \right\} dV = 0
\end{cases}
$$

$$
\begin{cases}
\sum_e \rho \left(\int_{V^e} N^{(\alpha)} N^{(\beta)} dV \right) \dfrac{\partial \boldsymbol{v}^{(\beta)}}{\partial t} + \sum_e \rho\, \overline{\boldsymbol{v}}^e \cdot \left(\int_{V^e} N^{(\alpha)} \nabla N^{(\beta)} dV \right) \boldsymbol{v}^{(\beta)} \\
\qquad - \sum_e \left(\int_{V^e} N^{(\alpha)} \nabla N^{(\beta)} dV \right) p^{(\beta)} + \sum_e \mu \left\{ \int_{V^e} \left(\nabla N^{(\alpha)} \cdot \nabla N^{(\beta)} \boldsymbol{I} + \nabla N^{(\beta)} \otimes \nabla N^{(\alpha)} \right) dV \right\} \cdot \boldsymbol{v}^{(\beta)} \\
\qquad + \sum_e \left\{ \int_{V^e} \tau\, \rho (\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\alpha)}) N^{(\beta)} dV \right\} \dfrac{\partial \boldsymbol{v}^{(\beta)}}{\partial t} + \sum_e \left\{ \int_{V^e} \tau\, \rho (\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\beta)})\ (\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\beta)}) dV \right\} \boldsymbol{v}^{(\beta)} \\
\qquad + \sum_e \left\{ \int_{V^e} \tau (\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\alpha)}) \nabla N^{(\beta)}\, dV \right\} p^{(\beta)} = \int_{S_t} N^{(\alpha)} \underline{\boldsymbol{t}}\, dS + \sum_e \rho \left(\int_{V^e} N^{(\alpha)} dV \right) \boldsymbol{b} + \sum_e \rho \left\{ \int_{V^e} \tau (\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\alpha)})\, dV \right\} \boldsymbol{b} \\
\sum_e \left(\int_{V^e} N^{(\alpha)} \nabla N^{(\beta)} dV \right) \cdot \boldsymbol{v}^{(\beta)} \\
\qquad + \sum_e \left(\int_{V^e} \tau_p \nabla N^{(\alpha)} N^{(\beta)} dV \right) \cdot \dfrac{\partial \boldsymbol{v}^{(\beta)}}{\partial t} + \sum_e \left\{ \int_{V^e} \tau_p \nabla N^{(\alpha)} (\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\beta)})\, dV \right\} \cdot \boldsymbol{v}^{(\beta)} \\
\qquad + \sum_e \left(\int_{V^e} \dfrac{\tau_p}{\rho} \nabla N^{(\alpha)} \cdot \nabla N^{(\beta)} dV \right) p^{(\beta)} = \sum_e \left(\int_{V^e} \tau_p \nabla N^{(\alpha)} dV \right) \cdot \boldsymbol{b}
\end{cases}
$$

SUPG/PSPG (8/10)

$$\begin{cases}
\sum_e \rho M^{(\alpha)(\beta)} \dfrac{\partial \boldsymbol{v}^{(\beta)}}{\partial t} + \sum_e \rho\left({}^n\overline{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\alpha)(\beta)}\right)\boldsymbol{v}^{(\beta)} - \sum_e \boldsymbol{C}^{(\alpha)(\beta)} p^{(\beta)} + \sum_e \mu\left\{\mathrm{tr}\boldsymbol{D}^{(\alpha)(\beta)}\boldsymbol{I} + (\boldsymbol{D}^{(\alpha)(\beta)})^{\mathrm{T}}\right\}\cdot \boldsymbol{v}^{(\beta)} \\
\quad + \sum_e \tau\, \rho(\overline{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\beta)(\alpha)}) \dfrac{\partial \boldsymbol{v}^{(\beta)}}{\partial t} + \sum_e \tau\, \rho\left\{({}^n\overline{\boldsymbol{v}}^e \otimes {}^n\overline{\boldsymbol{v}}^e) : \boldsymbol{D}^{(\alpha)(\beta)}\right\}\boldsymbol{v}^{(\beta)} + \sum_e \tau\,(\overline{\boldsymbol{v}}^e \cdot \boldsymbol{C}_v^{(\alpha)(\beta)})\, p^{(\beta)} \\
\quad = \int_{S_t} N^{(\alpha)} \underline{\boldsymbol{t}}\, dS + \sum_e \rho\left(\int_{V^e} N^{(\alpha)} dV\right)\boldsymbol{b} + \sum_e \rho\left\{\int_{V^e} \tau\,(\overline{\boldsymbol{v}}^e \cdot \nabla N^{(\alpha)})\, dV\right\}\boldsymbol{b} \\
\sum_e \boldsymbol{C}^{(\beta)(\alpha)} \cdot \boldsymbol{v}^{(\beta)} \\
\quad + \sum_e \tau_p \boldsymbol{M}_p^{(\alpha)(\beta)} \cdot \dfrac{\partial \boldsymbol{v}^{(\beta)}}{\partial t} + \sum_e \tau_p (\overline{\boldsymbol{v}}^e \cdot \boldsymbol{C}_v^{(\beta)(\alpha)}) \cdot \boldsymbol{v}^{(\beta)} + \sum_e \dfrac{\tau_p}{\rho} C_p^{(\alpha)(\beta)}\, p^{(\beta)} \\
\quad = \sum_e \left(\int_{V^e} \tau_p \nabla N_p^{(\alpha)} dV\right)\cdot \boldsymbol{b}
\end{cases}$$

τ に関する項
SUPG (Streamline Upwind Petrov Galerkin) 項

τ_p に関する項
PSPG (Pressure Stabilizing Petrov Galerkin)項

要素係数マトリックス

$$M^{(\alpha)(\beta)} = \int_{V^e} N^{(\alpha)} N^{(\beta)} dV \qquad \boldsymbol{A}^{(\alpha)(\beta)} = \int_{V^e} N^{(\alpha)} \nabla N^{(\beta)} dV \qquad \boldsymbol{C}^{(\alpha)(\beta)} = \boldsymbol{A}^{(\beta)(\alpha)} = \int_{V^e} \nabla N^{(\alpha)} N^{(\beta)}\, dV$$

$$\boldsymbol{D}^{(\alpha)(\beta)} = \int_{V^e} \nabla N^{(\alpha)} \otimes \nabla N^{(\beta)} dV \qquad \boldsymbol{C}_v^{(\alpha)(\beta)} = \boldsymbol{D}^{(\alpha)(\beta)} = \int_{V^e} \nabla N^{(\alpha)} \otimes \nabla N^{(\beta)}\, dV$$

$$\boldsymbol{M}_p^{(\alpha)(\beta)} = \boldsymbol{A}^{(\beta)(\alpha)} = \int_{V^e} \nabla N^{(\alpha)} N^{(\beta)} dV \qquad C_p^{(\alpha)(\beta)} = \mathrm{tr}\, \boldsymbol{D}^{(\alpha)(\beta)} = \int_{V^e} \nabla N^{(\alpha)} \cdot \nabla N^{(\beta)}\, dV$$

SUPG/PSPG (9/10)

速度に対して一般化Crank-Nicolson法

$$
\begin{cases}
\begin{aligned}
&\sum_e \left\{ \rho M^{(\alpha)(\beta)} + \tau\, \rho ({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\beta)(\alpha)}) \right\} \frac{{}^{n+1}\boldsymbol{v}^{(\beta)} - {}^n\boldsymbol{v}^{(\beta)}}{\Delta t} \\
&\quad + \sum_e \left[\rho ({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\alpha)(\beta)}) + \tau\, \rho \left\{ ({}^n\bar{\boldsymbol{v}}^e \otimes {}^n\bar{\boldsymbol{v}}^e) : \boldsymbol{D}^{(\alpha)(\beta)} \right\} \right] \left\{ \gamma\ {}^{n+1}\boldsymbol{v}^{(\beta)} + (1-\gamma)\ {}^n\boldsymbol{v}^{(\beta)} \right\} \\
&\quad - \sum_e \left\{ \boldsymbol{C}^{(\alpha)(\beta)} - \tau ({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{C}_v^{(\alpha)(\beta)}) \right\} {}^{n+1}p^{(\beta)} + \sum_e \mu \left\{ \mathrm{tr}\boldsymbol{D}^{(\alpha)(\beta)} \boldsymbol{I} + (\boldsymbol{D}^{(\alpha)(\beta)})^{\mathrm{T}} \right\} \cdot \left\{ \gamma\ {}^{n+1}\boldsymbol{v}^{(\beta)} + (1-\gamma)\ {}^n\boldsymbol{v}^{(\beta)} \right\} \\
&\quad = \int_{S_t} N^{(\alpha)} \underline{\boldsymbol{t}}\, dS + \sum_e \rho \left(\int_{V^e} N^{(\alpha)} dV \right) \boldsymbol{b} + \sum_e \rho \left\{ \int_{V^e} \tau ({}^n\bar{\boldsymbol{v}}^e \cdot \nabla N^{(\alpha)})\, dV \right\} \boldsymbol{b} \\
&\sum_e \boldsymbol{C}^{(\beta)\alpha} \cdot {}^{n+1}\boldsymbol{v}^{(\beta)} \\
&\quad + \sum_e \tau_p \boldsymbol{M}_p^{(\alpha)(\beta)} \cdot \frac{{}^{n+1}\boldsymbol{v}^{(\beta)} - {}^n\boldsymbol{v}^{(\beta)}}{\Delta t} + \sum_e \tau_p ({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{C}_v^{(\beta)(\alpha)}) \cdot \left\{ \gamma\ {}^{n+1}\boldsymbol{v}^{(\beta)} + (1-\gamma)\ {}^n\boldsymbol{v}^{(\beta)} \right\} + \sum_e \frac{\tau_p}{\rho} C_p^{(\alpha)(\beta)}\ {}^{n+1}p^{(\beta)} \\
&\quad = \sum_e \left(\int_{V^e} \tau_p \nabla N_p^{(\alpha)} dV \right) \cdot \boldsymbol{b}
\end{aligned}
\end{cases}
$$

$${}^n\bar{\boldsymbol{v}}^e = \frac{1}{V^e} \int_{V^e} {}^n\boldsymbol{v}\ dV$$ ：速度の要素平均値 [m/s]

通常のCrank-Nicolson法の場合 $\gamma = \dfrac{1}{2}$

SUPG/PSPG (10/10)

連立一次方程式 (求解には線形ソルバーが必要)

$$
\left\{
\begin{array}{l}
\displaystyle \sum_e \left[\frac{1}{\Delta t}\left\{\rho M^{(\alpha)(\beta)} + \tau\,\rho({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\beta)(\alpha)})\right\}\boldsymbol{I} + \gamma\left[\rho({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\alpha)(\beta)}) + \tau\,\rho\left\{({}^n\bar{\boldsymbol{v}}^e \otimes {}^n\bar{\boldsymbol{v}}^e):\boldsymbol{D}^{(\alpha)(\beta)}\right\}\right]\boldsymbol{I} + \gamma\,\mu\left\{\mathrm{tr}\boldsymbol{D}^{(\alpha)(\beta)}\boldsymbol{I} + (\boldsymbol{D}^{(\alpha)(\beta)})^{\mathrm{T}}\right\}\right]\cdot \underline{{}^{n+1}\boldsymbol{v}^{(\beta)}} \\
\displaystyle \quad - \sum_e \left\{\boldsymbol{C}^{(\alpha)(\beta)} - \tau({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{C}_v^{(\alpha)(\beta)})\right\} \underline{{}^{n+1}p^{(\beta)}} \\
\displaystyle \quad = \sum_e \left[\frac{1}{\Delta t}\left\{\rho M^{(\alpha)(\beta)} + \tau\,\rho({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\beta)(\alpha)})\right\}\boldsymbol{I} - (1-\gamma)\left[\rho({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{A}^{(\alpha)(\beta)}) + \tau\,\rho\left\{({}^n\bar{\boldsymbol{v}}^e \otimes {}^n\bar{\boldsymbol{v}}^e):\boldsymbol{D}^{(\alpha)(\beta)}\right\}\right]\boldsymbol{I} - (1-\gamma)\,\mu\left\{\mathrm{tr}\boldsymbol{D}^{(\alpha)(\beta)}\boldsymbol{I} + (\boldsymbol{D}^{(\alpha)(\beta)})^{\mathrm{T}}\right\}\right]\cdot {}^n\boldsymbol{v}^{(\beta)} \\
\displaystyle \quad + \int_{S_t} N^{(\alpha)}\underline{\boldsymbol{t}}\,dS + \sum_e \rho\left(\int_{V^e} N^{(\alpha)}dV\right)\boldsymbol{b} + \sum_e \rho\left\{\int_{V^e} \tau({}^n\bar{\boldsymbol{v}}^e \cdot \nabla N^{(\alpha)})\,dV\right\}\boldsymbol{b} \\
\displaystyle \sum_e \left[\boldsymbol{C}^{(\beta)(\alpha)} + \frac{1}{\Delta t}\tau_p \boldsymbol{M}_p^{(\alpha)(\beta)} + \gamma\,\tau_p({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{C}_v^{(\beta)(\alpha)})\right]\cdot \underline{{}^{n+1}\boldsymbol{v}^{\beta}} + \sum_e \frac{\tau_p}{\rho} C_p^{(\alpha)(\beta)}\, \underline{{}^{n+1}p^{\beta}} \\
\displaystyle \quad = \sum_e \left[\frac{1}{\Delta t}\tau_p \boldsymbol{M}_p^{\alpha\beta} - (1-\gamma)\,\tau_p({}^n\bar{\boldsymbol{v}}^e \cdot \boldsymbol{C}_v^{(\beta)(\alpha)})\right]\cdot {}^n\boldsymbol{v}^{\beta} \\
\displaystyle \quad + \sum_e \left(\int_{V^e} \tau_p \nabla N_p^{(\alpha)} dV\right)\cdot \boldsymbol{b}
\end{array}
\right.
$$

安定化パラメータ

$$
\left\{
\begin{array}{l}
\displaystyle \tau = \left[\left(\frac{2}{\Delta t}\right)^2 + \left(\frac{|{}^n\bar{\boldsymbol{v}}^e|}{h^e/2}\right)^2 + \left\{\frac{\mu/\rho}{(h^e/2)^2}\right\}^2\right]^{-\frac{1}{2}} \\
\tau_p = \tau
\end{array}
\right.
$$

h^e: 要素サイズ [m]

FrontISTRの非圧縮性流体解析機能
(機能説明および理論解説)

発表内容

1. 機能開発の目的
2. FrontISTR Ver. 4.6での機能
3. SUPG/PSPGによる定式化
4. 今後の展開

今後の展開

気液二相流，自然対流，Fluid-Structure Interaction，マルチフィジックスへ